

MATH 1104 LINEAR ALGEBRA

LECTURE NOTES

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(These Lecture Notes replace neither the Text Book nor the Lectures)

Part 7

- Inner Product, Length, and Orthogonality.
- Orthogonal Bases and Orthonormal Bases of R^n .
- Orthogonal Projection of one Vector onto Another.
- Orthogonal Projection of one Vector onto a Subspace of R^n .
- The Angle between two Vectors.

INNER PRODUCT, LENGTH, and ORTHOGONALITY

Definition: Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

Then the inner product of u and v is

$$u \cdot v = u^T v = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example: Let $u = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $v = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Then

$$u \cdot v = u^T v = \begin{bmatrix} 3 & -1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = 3 \cdot 6 + (-1) \cdot (-2) + (-5) \cdot 3 = 18 + 2 - 15 = 5,$$

and

$$v \cdot u = v^T u = \begin{bmatrix} 6 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 6 \cdot 3 + (-2)(-1) + 3 \cdot (-5) = 18 + 2 - 15 = 5.$$

So, $u \cdot v = v \cdot u$

Properties of the inner (dot) product

Let u , v , and w be vectors in R^n and c be a scalar. Then,

1. $u \cdot v = v \cdot u$
2. $(u + v) \cdot w = u \cdot w + v \cdot w$
3. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
4. $u \cdot u \geq 0$ and $u \cdot u = 0 \iff u = 0$.

Definition (the length of a vector):

If $v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$ in R^n , then the length (or norm) of v is defined by $\|v\| = \sqrt{v \cdot v}$.
Then, $\|v\|^2 = v \cdot v$.

A vector whose length is one unit is called a **unit vector**.

Example: Let $v = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. Find the length of v and a unit vector in the direction of v .

Solution:

$$\|v\|^2 = v \cdot v = 2^2 + (-3)^2 + 1^2 = 14,$$

$$\|v\| = \sqrt{14}$$

A unit vector in the direction of v is

$$u = \frac{v}{\|v\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -3/\sqrt{14} \\ 1/\sqrt{14} \end{bmatrix}.$$

The process of creating u from v is called **normalizing** v .

Example: Let $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

Find a unit vector v which is a basis for W .

Solution: let $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Then, $\|u\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$.

$$v = \frac{1}{\sqrt{6}}u = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

Another unit vector: $-v = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$.

Properties of length:

- (i) $\|u\| \geq 0$
- (ii) $\|u\| = 0 \iff u = 0$
- (iii) $\|ku\| = |k| \|u\|$, k is a scalar.
- (iv) $\|u + v\| \leq \|u\| + \|v\|$

Definition (Distance between u and v):

Let u, v in R^n . Then the distance between u and v is the length of the vector $u - v$. That is

$$\text{dist}(u, v) = \|u - v\|.$$

Example: Let $u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in R^3 .

$$\text{dist}(u, v) = \|u - v\| = \left\| \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

Properties of distance:

- (i) $\text{dist}(u, v) \geq 0$
- (ii) $\text{dist}(u, v) = \text{dist}(v, u)$
- (iii) $\text{dist}(u, v) = 0 \iff u = v$.
- (iv) $\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$

Definition (Orthogonal vectors): Let u and v be two vectors in R^n . Then u and v are orthogonal to each other ($u \perp v$) if $u \cdot v = 0$.

Note: Zero vector is orthogonal to every vector.

Example (The Pythagorean Theorem): Let u and v in R^n . Show that

$$u \perp v \iff \|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Solution:

$$\|u + v\|^2 = (u + v) \cdot (u + v)$$

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v \quad (*)$$

Then the result follows from (*).

Exercise: Let u and v in R^n . Show that

$$u \perp v \iff \|u + v\|^2 = \|u - v\|^2.$$

Hint: Consider the equations

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v \text{ and } \|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2u \cdot v$$

Question: Does

$$\|u + v + z\|^2 = \|u\|^2 + \|v\|^2 + \|z\|^2$$

imply that $\{u, v, z\}$ is an orthogonal set?

Answer: No. Take $u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $z = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Then,

$$\|u + v + z\|^2 = \left\| \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\|^2 = 6,$$

$$\|u\|^2 = \|v\|^2 = \|z\|^2 = 2, \|u + v + z\|^2 = \|u\|^2 + \|v\|^2 + \|z\|^2.$$

But $\{u, v, z\}$ is not an orthogonal set.

Definition: Let W be a subspace of R^n .

1. If $z \in R^n$ is orthogonal to every vector in W , then z is said to be orthogonal to W .
2. The set of all z that are orthogonal to W is called the orthogonal complement of W , and is denoted by W^\perp . That is ,

$$W^\perp = \{z \in R^n \mid z \perp W\}.$$

Example: Let $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Then, $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \perp W$ and $W^\perp = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Example: Let $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Then, $z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \perp W$ and $W^\perp = \text{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Example: Let $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $S = \{u_1, u_2\}$. Find S^\perp .

Solution:

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S^\perp \iff v \cdot u_1 = 0 \text{ and } v \cdot u_2 = 0.$$

So,

$$x + 2y + z = 0 \text{ and } x - y - z = 0.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 0 \end{array} \right] \implies v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}z \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

$$S^\perp = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$$

Example: Let $b = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. Find two linearly independent vectors which are orthogonal to b .

Solution:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \perp \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \iff x_1 - 2x_2 + 4x_3 = 0.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $u = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent and

they are both perpendicular to the vector b . In fact, $b \perp \text{Span}\{u, v\}$.

Exercise: Show that if $y \perp u$ and $y \perp v \implies y \perp (u + v)$.

Note: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \not\perp \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \not\perp \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.

Theorem:

- i. A vector x is in W^\perp if and only if x is orthogonal to every vector in a set that spans W .
- ii. W^\perp is a subspace of R^n .

Remark: If x is in both W and W^\perp , then $x = 0$.

Proof: If $x \in W$ and $x \in W^\perp$, then $x \perp x$. Thus $x \cdot x = 0 \implies x = 0$.

$$W \cap W^\perp = \{0\}$$

Theorem: Let A be an $m \times n$ matrix. Then,

$$(i) (\text{Row } A)^\perp = \text{Nul } A \quad (ii) (\text{Col } A)^\perp = \text{Nul } A^T$$

Proof (ii): In (i), replacing A by A^T gives $(\text{Row } A^T)^\perp = \text{Nul } A^T$,
i.e. $(\text{Col } A)^\perp = \text{Nul } A^T$.

Example: Let $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$. Find $(\text{Col } A)^\perp$.

Solution: Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^2$.

$$\left. \begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \perp \begin{bmatrix} 3 \\ 6 \end{bmatrix} &\iff 3x_1 + 6x_2 = 0. \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \perp \begin{bmatrix} 2 \\ 4 \end{bmatrix} &\iff 2x_1 + 4x_2 = 0. \end{aligned} \right\} \implies x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$$(\text{Col } A)^\perp = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

- A set of non-zero vectors S is called an **orthogonal set** if each vector in S is orthogonal to other vectors in S , i.e.,

$$S = \{v_1, v_2, \dots, v_p\} \text{ is an orthogonal set } \iff v_i \cdot v_j = 0 \text{ if } i \neq j.$$

- An orthogonal set in which each vector has length 1 is called an **orthonormal set**.
- A basis consisting of orthogonal vectors is called an **orthogonal basis**.
- A basis consisting of orthonormal vectors is called an **orthonormal basis**.

An orthogonal set in R^3 : $\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}$

An orthonormal set in R^3 : $\left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}$

Orthogonal bases for R^3 :

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

Orthonormal bases for R^3 :

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{B}_3 = \left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \right\},$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{12} \\ 2/\sqrt{12} \\ 2/\sqrt{12} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} \right\}.$$

An orthogonal set in R^4 : $\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$

An orthogonal basis for R^4 : $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

An orthonormal set in R^4 : $\left\{ \begin{bmatrix} 2/\sqrt{24} \\ 2/\sqrt{24} \\ 4/\sqrt{24} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}.$

An orthonormal basis for R^4 : $\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

Definition:

An **orthogonal matrix** is a square matrix with orthonormal columns.

Theorem:

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Proof: For simplicity, suppose that U has only three columns, each a vector in R^m . Then, $U = [u_1 \ u_2 \ u_3]_{m \times 3}$ and

$$U^T U = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}_{3 \times m} [u_1 \ u_2 \ u_3]_{m \times 3} = \begin{bmatrix} u_1^T u_1 & u_1^T u_2 & u_1^T u_3 \\ u_2^T u_1 & u_2^T u_2 & u_2^T u_3 \\ u_3^T u_1 & u_3^T u_2 & u_3^T u_3 \end{bmatrix}_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}.$$

Thus, an **orthogonal matrix** is a square invertible matrix U such that $U^{-1} = U^T$.

Remark: Let $U = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}.$

U is a non-square matrix with orthonormal columns.

$$U^T U = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$U U^T = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 5/9 & -4/9 & 2/9 \\ -4/9 & 5/9 & 2/9 \\ 2/9 & 2/9 & 8/9 \end{bmatrix} \neq I.$$

Example: Let $U = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$.

U is a square matrix with orthonormal columns. Then, rows of U are orthonormal as well.

$$U^T U = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U U^T = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U U^T = U^T U = I \implies U^{-1} = U^T.$$

Example: Let $U = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$.

U is an orthogonal matrix $\implies U^T = U \implies U^T U = U U = I \implies U^{-1} = U$.

Example: The columns of the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

are orthogonal but not orthonormal.

Hence, A is not an orthogonal matrix.

Note that $A^T A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix but not the identity matrix.

Theorem: Let U be an $m \times n$ matrix with orthonormal columns, and let x and y be in R^n . Then,

(a) $\|Ux\| = \|x\|$

(b) $(Ux) \cdot (Uy) = x \cdot y$

(c) $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$.

Proof:

(a) $\|Ux\|^2 = Ux \cdot Ux = (Ux)^T(Ux) = (x^T U^T)(Ux) = x^T Ix = \|x\|^2.$

(b) $Ux \cdot Uy = (Ux)^T(Uy) = (x^T U^T)(Uy) = x^T Iy = x \cdot y.$

Part (c) follow from part (b).

Remark: The linear transformation $T(x) = Ux$ preserves lengths and orthogonality.

Theorem: If $S = \{u_1, u_2, \dots, u_p\}$ is an orthogonal set of nonzero vectors in R^n , then S is linearly independent.

Proof: Let $0 = c_1 u_1 + c_2 u_2 + \dots + c_p u_p$ for some scalars c_1, c_2, \dots, c_p . Then,

$$0 \cdot u_1 = (c_1 u_1 + c_2 u_2 + \dots + c_p u_p) \cdot u_1$$

$$0 = c_1 u_1 \cdot u_1 + c_2 u_2 \cdot u_1 + \dots + c_p u_p \cdot u_1,$$

$$0 = c_1 u_1 \cdot u_1 + 0 + \dots + 0, \quad (\text{since } u_i \cdot u_j = 0 \text{ if } i \neq j).$$

$$0 = c_1 \|u_1\|^2 \implies c_1 = 0 \quad (u_1 \neq 0).$$

Similarly, we can show that $c_2 = 0, \dots, c_p = 0$. Thus, S is linearly independent.

Theorem: Let $\{u_1, u_2, \dots, u_n\}$ be an orthogonal basis for R^n . Then for each $x \in R^n$,

$$x = c_1 u_1 + c_2 u_2 + \dots + c_n u_n, \quad \text{where } c_i = \frac{x \cdot u_i}{u_i \cdot u_i}, \quad i = 1, 2, \dots, n.$$

Proof: For a fixed i , $1 \leq i \leq n$,

$$x \cdot u_i = (c_1 u_1 + c_2 u_2 + \cdots + c_n u_n) \cdot u_i = c_i (u_i \cdot u_i),$$

which gives

$$c_i = \frac{x \cdot u_i}{u_i \cdot u_i} \quad (\text{since } u_i \neq 0, u_i \cdot u_i \neq 0).$$

Example: Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and

$S = \{u_1, u_2, u_3\}$. Express x as a linear combination of the vectors in S .

Solution: Clearly $u_i \cdot u_j = 0$, when $i \neq j$.

Thus, S is an orthogonal basis for R^3 and

$$x = c_1 u_1 + c_2 u_2 + c_3 u_3,$$

where

$$c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{9}{9} = 1, \quad c_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{6}{9} = \frac{2}{3}, \quad c_3 = \frac{x \cdot u_3}{u_3 \cdot u_3} = \frac{3}{9} = \frac{1}{3}.$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

Orthogonal Projection of one Vector Onto Another

Let u be a non-zero vector in R^n and let y be a vector in R^n .

We can decompose y into a sum

$$y = \hat{y} + z,$$

where

$$\hat{y} = \text{proj}_u y = \frac{y \cdot u}{u \cdot u} u \quad \text{and} \quad z = y - \hat{y}.$$

$\hat{y} \in \text{Span}\{u\}$ and $z \perp u$.

\hat{y} is “the orthogonal projection of y onto u ”

z is “the component of y is orthogonal to u .”

$$y = \alpha u + z \quad (*)$$

$$z = y - \alpha u$$

$$z \perp u \implies y - \alpha u \perp u \implies (y - \alpha u) \cdot u = 0.$$

Thus,

$$y \cdot u - (\alpha u) \cdot u = 0$$

$$y \cdot u - \alpha(u \cdot u) = 0$$

$$\alpha = \frac{y \cdot u}{u \cdot u} \quad (**)$$

(*) and (**) gives $y = \frac{y \cdot u}{u \cdot u} u + z$.

Clearly $\hat{y} = \alpha u$ and $z \perp u$.

$$\hat{y} = \text{proj}_u y = \frac{y \cdot u}{u \cdot u} u$$

Let $L = \text{Span}\{u\}$. Then, $\text{proj}_u y = \text{proj}_L y$.

Example: Let $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

Write y as the sum of a vector in $\text{Span}\{u\}$ and a vector orthogonal to u .

Solution:

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{20}{50} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} \in \text{Span}\{u\}.$$

$$z = y - \hat{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix} \perp u.$$

$$y = \hat{y} + z \implies \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}.$$

Example: Let $y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$.

Write y as the sum of a vector in $\text{Span}\{u\}$ and a vector orthogonal to u .

Solution:

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{7}{25} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 21/25 \\ 28/25 \end{bmatrix}.$$

$$z = y - \hat{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 21/25 \\ 28/25 \end{bmatrix} = \begin{bmatrix} 1 \\ 4/25 \\ -3/25 \end{bmatrix}.$$

$$y = \hat{y} + z \implies \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 21/25 \\ 28/25 \end{bmatrix} + \begin{bmatrix} 1 \\ 4/25 \\ -3/25 \end{bmatrix}.$$

Example: Let $x = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Compute the distance from x to the line through u and origin.

Solution:

$$\hat{x} = \frac{x \cdot u}{u \cdot u} u = \frac{15}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

The distance is

$$\begin{aligned} \text{dist}(x, \hat{x}) &= \|x - \hat{x}\| = \left\| \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -6 \\ 3 \end{bmatrix} \right\| \\ &= \sqrt{(-6)^2 + 3^2} = \sqrt{45} = 3\sqrt{5}. \end{aligned}$$

Orthogonal Projection of one Vector onto a Subspace of R^n

Definition: Let W be a subspace of R^n with an orthogonal basis $\{u_1, u_2, \dots, u_p\}$ and let x be a vector in R^n .

The orthogonal projection of x onto W is

$$\begin{aligned} \text{proj}_W x &= \text{proj}_{u_1} x + \text{proj}_{u_2} x + \cdots + \text{proj}_{u_p} x \\ &= \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{x \cdot u_p}{u_p \cdot u_p} u_p. \end{aligned}$$

Theorem: Let W be a subspace of R^n .

Then each vector $x \in R^n$ can be written uniquely in the form

$$x = \hat{x} + z,$$

where $\hat{x} = \text{proj}_W x \in W$ and $z = x - \hat{x} \in W^\perp$.

$x = \hat{x} + z$ is called the **orthogonal decomposition** of x onto W .

Example: Let $x = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $W = \text{Span}\{u_1, u_2\}$.
Find the orthogonal decomposition of x onto W .

Solution: $\{u_1, u_2\}$ is an orthogonal basis for W .

$$\hat{x} = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{12}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \in W.$$

$$z = x - \hat{x} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in W^\perp.$$

$$x = \hat{x} + z \implies \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Example: Let $y = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$, $u_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, and $u_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$. Write y as the sum of a vector in $W = \text{span}\{u_1, u_2\}$ and a vector orthogonal to W .

Solution:

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{30}{25} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \frac{15}{25} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} \in W.$$

$$z = y - \hat{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \in W^\perp.$$

$$y = \hat{y} + z \implies \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

Example: Let $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ and $W = \text{Span}\{u_1, u_2\}$.

a) Find the orthogonal projection of y onto W .

b) Write y as the sum of a vector in W and a vector orthogonal to W .

Soln: a) $\{u_1, u_2\}$ is an orthogonal basis for W .

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}.$$

b)

$$z = y - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}.$$

$$y = \hat{y} + z, \text{ where } \hat{y} \in W \text{ and } z \in W^\perp.$$

The Best Approximation Theorem:

Let W be a subspace of R^n , and y be a vector in R^n . Let \hat{y} be the orthogonal projection of y onto W . Then, \hat{y} is the closest point in W to y , i.e,

$$\|y - \hat{y}\| < \|y - v\|$$

for all v in W distinct from \hat{y} .

Definition (The distance from a point to a subspace):

The distance from a point y in R^n to a subspace W is defined as the distance from y to the nearest point in W .

$$\text{dist}(x, W) = \text{dist}(x, \hat{x}) = \|x - \hat{x}\|.$$

Example: Let $x = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ and $W = \text{Span}\{u_1, u_2\}$.

a) Find the orthogonal projection of x onto W .

b) What is the nearest point in W to x ?

c) Find the distance from x to W .

d) Find an orthonormal basis for W .

Solution: a)

$$\hat{x} = \text{proj}_W x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{-2}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \frac{2}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

b) The nearest point in W to x is $\hat{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

c) The distance from x to W :

$$\text{dist}(x, W) = \text{dist}(x, \hat{x}) = \|x - \hat{x}\| = \left\| \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\| = \sqrt{16} = 4.$$

d) An orthonormal basis for W is $\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$.

The Angle between two Vectors

The angle between two non-zero vectors u and v is

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}, \quad 0 \leq \theta \leq \pi.$$

Proof in R^2 : Consider the vectors $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and the triangle

with sides of length $\|u\|$, $\|v\|$ and $\|u - v\|$. By the law of cosines,

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta.$$

Thus,

$$\begin{aligned} \|u\| \|v\| \cos \theta &= \frac{1}{2} \left[\|u\|^2 + \|v\|^2 - \|u - v\|^2 \right] \\ &= \frac{1}{2} \left[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 \right] \\ &= u_1 v_1 + u_2 v_2 = u \cdot v \end{aligned}$$

Proof is similar for vectors in R^3 . Formula may be used to define the angle between two vectors in R^n for $n > 3$.

Example: Find the angle θ between the pairs of vectors in each part:

a) $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$:

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{3}{\sqrt{2}\sqrt{9}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}.$$

b) $u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$:

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{1}{\sqrt{4}\sqrt{1}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

$$\text{c) } u = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} :$$

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{-2}{\sqrt{4}\sqrt{4}} = \frac{-1}{2} \implies \theta = \frac{2\pi}{3}.$$

$$\text{d) } u = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} :$$

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{0}{\sqrt{15}\sqrt{21}} = 0 \implies \theta = \frac{\pi}{2}.$$